Theorem:
The potential energy U of an extended object is equal to the product of the total mass M, the gravitational constant g, and the y-coordinate of the center of mass, y_{cm}.

$$U = Mgy_{\text{cm}}$$

It does not matter how the object is oriented as long as the height of the center of mass is known.

Proof:
Suppose we have an object (such as a rod) of length L whose with density $\lambda(y)$ where y is some linear dimension. We can divide the object into an infinite number of infinitesimally thin cross sections each of which contributes a potential energy equal to

$$\mathrm{d}U = gy\mathrm{d}m \qquad \text{(Equation 1)}$$

Since $\lambda(y) = \frac{\mathrm{d}m}{\mathrm{d}y}$, we can substitute $\mathrm{d}m = \lambda(y)\mathrm{d}y$, resulting in

$$\mathrm{d}U = gy\lambda(y)\mathrm{d}y \qquad \text{(Equation 2)}$$

We compute the overall energy by integrating these energies from y = 0 to y = L

$$U = \int_0^L gy\lambda(y)\mathrm{d}y$$

$$U = g\int y\lambda(y)\mathrm{d}y \qquad \text{(Equation 3)}$$

Meanwhile, the center of mass of the object is located at

$$y_{\text{cm}} = \frac{\int_0^L y\lambda(y)dy}{\int_0^L \lambda(y)dy}$$

$$y_{\text{cm}} = \frac{1}{M}\int_0^L y\lambda(y)dy \qquad \text{(Equation 4)}$$

Solving Equation (4) for the integral, and substituting into Equation (3), we arrive at

$$\boxed{U = Mgy_{\text{cm}}} \qquad \text{(Equation 5)}$$

This equation means that the potential energy depends only on the total mass and the height of the center of mass—but not on orientation of the object relative to its center of mass.

Example:

Suppose you have a thin rod of length L whose density is proportional to the distance from one end according to $\lambda(y) = ky$ where k is a constant.

The total mass of the rod is

$$M = \int_0^L \mathrm{d}m = \int_0^L \lambda \mathrm{d}y = \int_0^L ky\mathrm{d}y = \frac{1}{2}kL^2 \qquad \text{(Equation 1)}$$

According to Equation 4, the center of mass of the rod is located at

$$\frac{\int_0^L y\mathrm{d}m}{M} = \frac{\int_0^L y\lambda\mathrm{d}y}{\frac{1}{2}kL^2} = \frac{\int_0^L ky^2\mathrm{d}y}{\frac{1}{2}kL^2} = \frac{\frac{1}{3}kL^3}{\frac{1}{2}kL^2} = \frac{2}{3}L \qquad \text{(Equation 2)}$$

If this rod is orientated so that all of its mass is located at a height of L, then its potential energy is

$$U = Mgy_{\mathrm{cm}} = \left(\frac{1}{2}kL^2\right)gL = \frac{1}{2}kgL^3 \qquad \text{(Equation 3)}$$

If the rod is orientated upright so that the least dense end is touching the ground, then its $y_{\mathrm{cm}} = \frac{2}{3}L$, so

$$U = Mgy_{\mathrm{cm}} = \left(\frac{1}{2}kL^2\right)g\left(\frac{2}{3}L\right) = \frac{1}{3}kgL^3 \qquad \text{(Equation 4)}$$

This is 2/3 of the maximum potential energy.

If the rod is inverted so its densest end touches the ground, then $y_{\mathrm{cm}} = \dfrac{1}{3}L$, so

$$U = Mgy_{\mathrm{cm}} = \left(\frac{1}{2}kL^2\right)g\left(\frac{1}{3}L\right) = \frac{1}{6}kgL^3 \qquad \text{(Equation 5)}$$

If the rod is orientated with its least dense end touching the ground and tilted at angle of θ, then all of its component masses are reduced in height by a factor of $\sin\theta$ so y_{cm} becomes $\frac{2}{3}L\sin\theta$. Therefore,

$$U = Mgy_{\mathrm{cm}} = \left(\frac{1}{2}kL^2\right)g\left(\frac{2}{3}L\sin\theta\right) = \frac{1}{3}kgL^3\sin\theta \qquad \text{(Equation 6)}$$

If the rod is orientated with its least dense end touching the ground and tilted at angle of θ, then all of its component masses are reduced in height by a factor of $\sin\theta$ so y_{cm} becomes $\frac{1}{3}L\sin\theta$. Therefore,

$$U = Mgy_{\mathrm{cm}} = \left(\frac{1}{2}kL^2\right)g\left(\frac{1}{3}L\sin\theta\right) = \frac{1}{6}kgL^3\sin\theta \qquad \text{(Equation 7)}$$